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On a theory of nonlocal elasticity of bi-Helmholtz type and some applications

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Abstract

A theory of nonlocal elasticity of bi-Helmholtz type is studied. We employ Eringen's model of nonlocal elasticity, with bi-Helmholtz type kernels, to study dispersion relations, screw and edge dislocations. The nonlocal kernels are derived analytically as Green functions of partial differential equations of fourth order. This continuum model of nonlocal elasticity involves two material length scales which may be derived from atomistics. The new nonlocal kernels are nonsingular in one-, two- and three-dimensions. Furthermore, the nonlocal elasticity of bi-Helmholtz type improves the one of Helmholtz type by predicting a dispersion relationship with zero group velocity at the end of the first Brillouin zone. New solutions for the stresses and strain energy of screw and edge dislocations are found.

Keywords: Nonlocal elasticity; Dislocations

1. Introduction

Classical continuum theories like the linear theory of elasticity are intrinsically size independent. The classical theory of elasticity predicts no dispersion and is valid only for small wave numbers. In addition, the elastic strain, the stress and the elastic strain energy of defects (dislocations, disclinations) are singular

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at the defect line. Of course, such singularities are unphysical and they are the price which has to be paid if one uses classical elasticity within the defect core region.

An improvement is obtained by using nonlocal elasticity instead of classical one (Kröner and Datta, 1966; Eringen, 1983; Eringen, 2002). Theory of nonlocal elasticity includes the effect of long range interatomic forces so that it can be used as a continuum model of the atomic lattice dynamics. In the theory of nonlocal elasticity the stress at a reference point $\boldsymbol{r}$ depends on the elastic strain at every point $\boldsymbol{r}'$. From the mathematical point of view, this nonlocal interaction is given by a so-called nonlocal kernel. Solutions for the screw and edge dislocations within nonlocal elasticity with Gaussian kernels have been given by Eringen (1977a,b). A feature of these solutions is the elimination of the stress field and strain energy singularities at the dislocation line. For a special class of kernels, which are the Green functions of the Helmholtz equation, nonlocal elasticity was studied by Eringen (1983, 1992, 2002). These kernel are singular in two- and three-dimensions. This nonlocal elasticity of Helmholtz type was used for the calculation of the stress field and strain energy of a screw dislocation. Fortunately, the singularities of the stress disappeared. This stress field of a screw dislocation coincides with the stress field calculated by Gutkin and Aifantis (1999a) within the gradient elasticity framework. Furthermore, the strain energy is finite at the dislocation core. The stress field of an edge dislocation was calculated by Lazar (2003) in the framework of nonlocal elasticity of Helmholtz type. His result is in agreement with the stress obtained by Gutkin and Aifantis (1999a) within gradient elasticity. On the other hand, the predicted dispersion curve is more realistic in nonlocal elasticity than that obtained within the classical elasticity. However, the group velocity is badly off at the end of the Brillouin zone. These solutions for the stress fields converge to the linear elasticity solutions in the far field.

Thus, the question arises, how can we improve the nonlocal elasticity of Helmholtz type? The kernels should be the Green functions of partial differential equations of higher order which are modifications of the Helmholtz equation. The physical motivation of this paper is twofold. First, the nonlocal kernels should be nonsingular and, thus, they should have a finite maximum value. Second, the kernels of bi-Helmholtz type should lead to dispersion relations for plane harmonic waves, which are coincident to those obtained in lattice dynamics. Thus, the group velocity must be zero at the boundary of the Brillouin zone. The physical reason is that the limit-velocity with $k = \pi/a$ belongs to a standing wave (no wave propagation)–a well-known result in lattice dynamics. Recently, Picu (2002) discussed the implications of modified Gaussian kernels which depends on two intrinsic length scales to improves the behaviour of the group velocity.

In this paper, we address the theory of nonlocal elasticity of bi-Helmholtz type. In Section 2, the framework of isotropic nonlocal elasticity is discussed. We specialize to nonlocal elasticity of bi-Helmholtz type in Section 3. In Section 4, we give the nonlocal kernels of such a theory in one-, two- and three-dimensions. In Section 5, we give the dispersion relations and compare them with results obtained in lattice theories. The stresses and strain energies of screw and edge dislocations are calculated in Sections 6 and 7. Finally, we conclude with a summary in Section 8. Some mathematical details are given in Appendix A.

2. Governing equations

The fundamental field equations of nonlocal elasticity have for an isotropic medium the following form (Eringen, 1983, 1987, 2002):

$$\partial_j t_{ij} + \rho(f_i - \ddot{u}_i) = 0, \tag{2.1}$$

$$t_{ij}(\boldsymbol{r}) = \int_V \alpha(|\boldsymbol{r} - \boldsymbol{r}'|)\sigma_{ij}(\boldsymbol{r}')\mathrm{d}v(\boldsymbol{r}'), \qquad t_{ij} = t_{ji} \tag{2.2}$$

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}, \tag{2.3}$$

$$e_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \tag{2.4}$$

where λ and μ are the Lamé constants. ρ, f_i and u_i are, respectively, the mass density, body force density and the displacement vector. In addition, e_{ij} is the classical elastic strain tensor, σ_{ij} and t_{ij} are the classical and nonlocal stress tensors, respectively. The $\alpha(|\boldsymbol{r}-\boldsymbol{r}'|)$ is the nonlocal kernel. Boundary conditions involving tractions are based on the stress tensor t_{ij},

$$t_{ij}n_j = t_{i(n)}, \tag{2.5}$$

where $t_{i(n)}$ are the boundary tractions. Eq. (2.2) may be rewritten in a nonlocal stress-strain relation according to

$$t_{ij}(\boldsymbol{r}) = \int_V \{\lambda'(|\boldsymbol{r}-\boldsymbol{r}'|)\delta_{ij}\delta_{kl} + 2\mu'(|\boldsymbol{r}-\boldsymbol{r}'|)\delta_{ik}\delta_{jl}\}e_{kl}(\boldsymbol{r}')\,\mathrm{d}v(\boldsymbol{r}') \tag{2.6}$$

with

$$\lambda'(|\boldsymbol{r}-\boldsymbol{r}'|) = \lambda\alpha(|\boldsymbol{r}-\boldsymbol{r}'|), \quad \mu'(|\boldsymbol{r}-\boldsymbol{r}'|) = \mu\alpha(|\boldsymbol{r}-\boldsymbol{r}'|), \tag{2.7}$$

which are the Lamé coefficients of the nonlocal medium. Thus, the nonlocal kernel is a measure of the effect of the strain at $\boldsymbol{r}'$ on the stress at $\boldsymbol{r}$.

The nonlocal kernel $\alpha(|\boldsymbol{r}-\boldsymbol{r}'|)$ has the following properties:

(i) From Eq. (2.2) it is clear that the nonlocal kernel has the dimension of $(\text{length})^{-3}$. Therefore, it must depend on characteristic length scales.
(ii) It must reach a maximum at $\boldsymbol{r}=\boldsymbol{r}'$ and has to decay to zero at large distances.
(iii) It has to satisfy the normalization condition:

$$\int_V \alpha(|\boldsymbol{r}-\boldsymbol{r}'|)\,\mathrm{d}v(\boldsymbol{r}') = 1, \tag{2.8}$$

which is the normalization condition of the nonlocal kernel.
(iv) It must be a continuous function of position (in the classical limit it becomes the Dirac delta function).These rather general conditions can be fulfilled by many functions. An additional property of the nonlocal kernel may be:
(v) It is a Green function of a linear differential operator L:

$$L\alpha(|\boldsymbol{r}-\boldsymbol{r}'|) = \delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{2.9}$$

L may be a differential operator with constant or variable coefficients of any order. Applying the differential operator L to Eq. (2.2), we obtain the differential equation for t_{ij}:

$$Lt_{ij} = \sigma_{ij}, \tag{2.10}$$

where the inhomogeneous part is given by the stress tensor σ_{ij}. If L is a differential operator with constant coefficients, then (2.1) gives

$$\partial_j\sigma_{ij} + L(\rho f_i - \rho\ddot{u}_i) = 0. \tag{2.11}$$

3. Nonlocal elasticity of bi-Helmholtz type

As elliptic differential operator we use an operator of fourth order which has already been proposed by Eringen (1992, 2002). This linear differential operator of fourth order of bi-Helmholtz type with constant coefficients is given by

$$L = 1 - \varepsilon^2\Delta + \gamma^4\Delta\Delta, \tag{3.1}$$

where ε and γ are nonnegative parameters of nonlocality. These two parameters have the dimension of lengths and, thus, they may be written in terms of a characteristic length scale, a, (i.e., lattice constant)

$$\varepsilon = \varepsilon_0 a, \quad \varepsilon_0 \geqslant 0, \quad \gamma = \gamma_0 a, \quad \gamma_0 \geqslant 0. \tag{3.2}$$

Nonlocal elasticity of bi-Helmholtz type has two limits. They are given in two steps. The limit from bi-Helmholtz type to Helmholtz type is given by $\gamma \to 0$. The second limit is obtained by $\varepsilon \to 0$, in addition to the first one. The second limit is the limit from nonlocal elasticity of Helmholtz type to classical theory of elasticity.

Using (3.1), Eq. (2.9) reads

$$(1 - \varepsilon^2 \Delta + \gamma^4 \Delta\Delta)\alpha(|\boldsymbol{r} - \boldsymbol{r}'|) = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{3.3}$$

and (2.10) gives

$$(1 - \varepsilon^2 \Delta + \gamma^4 \Delta\Delta)t_{ij} = \sigma_{ij}. \tag{3.4}$$

In addition, Eq. (2.11) is given by

$$\partial_j \sigma_{ij} + (1 - \varepsilon^2 \Delta + \gamma^4 \Delta\Delta)(\rho f_i - \rho \ddot{u}_i) = 0. \tag{3.5}$$

Using Eqs. (2.3) and (2.4), we obtain from (3.5) the following partial differential equation for the displacement vector:

$$\mu \Delta u_i + (\lambda + \mu)\partial_i \partial_j u_j + (1 - \varepsilon^2 \Delta + \gamma^4 \Delta\Delta)(\rho f_i - \rho \ddot{u}_i) = 0. \tag{3.6}$$

Because Eq. (3.6) involves mixed space-time derivatives up to fourth order space derivatives, it is an improved linear version of the 'good' Boussinesq equation (see, e.g., Maugin, 1999).

The differential operator is called bi-Helmholtz operator because it can be factorized into two Helmholtz operators according to

$$L = 1 - \varepsilon^2 \Delta + \gamma^4 \Delta\Delta = (1 - c_1^2 \Delta)(1 - c_2^2 \Delta) \tag{3.7}$$

with the abbreviations

$$c_1^2 = \frac{\varepsilon^2}{2}\left(1 + \sqrt{1 - 4\frac{\gamma^4}{\varepsilon^4}}\right), \tag{3.8}$$

$$c_2^2 = \frac{\varepsilon^2}{2}\left(1 - \sqrt{1 - 4\frac{\gamma^4}{\varepsilon^4}}\right), \tag{3.9}$$

and the conditions

$$\varepsilon^2 = c_1^2 + c_2^2, \tag{3.10}$$

$$\gamma^4 = c_1^2 c_2^2. \tag{3.11}$$

We note that such a factorization was not used by Eringen (2002). Due to (3.11), the condition of the discriminant in Eqs. (3.8) and (3.9) must be nonnegative. Thus,

$$0 \leqslant \left(1 - 4\frac{\gamma^4}{\varepsilon^4}\right). \tag{3.12}$$

Then, we have the two possibilities:

- $\varepsilon^4 > 4\gamma^4$, $c_1 \neq c_2$ are real, $\Rightarrow \ \varepsilon > \sqrt{2}\gamma$.
- $\varepsilon^4 = 4\gamma^4$, $c_1 = c_2$ are real, $\Rightarrow \ \varepsilon = \sqrt{2}\gamma$.

Since c_2^2 is positive we have

$$0 \leqslant 4\gamma^4 \leqslant \varepsilon^4. \tag{3.13}$$

Thus, c_1^2 and c_2^2 have to be real and positive. The limit from nonlocal elasticity of bi-Helmholtz type to nonlocal elasticity of Helmholtz type is obtained from Eqs. (3.8) and (3.9) by $c_1^2 \to \varepsilon^2$ and $c_2^2 \to 0$.

4. Nonlocal kernels of bi-Helmholtz type

We give the solutions of Eq. (3.3) for infinitely extended solids. We note that Eringen (2002) gave only the expression (A.4) for the kernel of Eq. (3.3) in the k-space. The explicit calculations are given in Appendix A.1. The nonlocal kernels which are the Green function of the bi-Helmholtz equation are given by (see Eqs. (A.12)–(A.14)):

(a) One-dimension ($|\boldsymbol{r}| = \sqrt{x^2}$):

$$\alpha(|\boldsymbol{r}|) = \frac{1}{2}\frac{1}{c_1^2 - c_2^2}\{c_1 \exp(-|\boldsymbol{r}|/c_1) - c_2 \exp(-|\boldsymbol{r}|/c_2)\}, \quad \alpha(0) = \frac{1}{2}\frac{1}{c_1 + c_2}. \tag{4.1}$$

(b) Two-dimensions ($|\boldsymbol{r}| = \sqrt{x^2 + y^2}$):

$$\alpha(|\boldsymbol{r}|) = \frac{1}{2\pi}\frac{1}{c_1^2 - c_2^2}\{K_0(|\boldsymbol{r}|/c_1) - K_0(|\boldsymbol{r}|/c_2)\}, \quad \alpha(0) = \frac{1}{2\pi}\frac{1}{c_1^2 - c_2^2}\ln\frac{c_1}{c_2}, \tag{4.2}$$

where K_n is the modified Bessel function of the second kind (or McDonald function) of order n.

(c) Three-dimensions ($|\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$):

$$\alpha(|\boldsymbol{r}|) = \frac{1}{4\pi}\frac{1}{c_1^2 - c_2^2}\frac{1}{|\boldsymbol{r}|}\{\exp(-|\boldsymbol{r}|/c_1) - \exp(-|\boldsymbol{r}|/c_2)\}, \quad \alpha(0) = \frac{1}{4\pi}\frac{1}{c_1 c_2 (c_1 + c_2)} \tag{4.3}$$

and for the limit $c_2 \to c_1 = \gamma$ we obtain

(a) One-dimension:

$$\alpha(|\boldsymbol{r}|) = \frac{1}{2}\frac{1}{2\gamma^2}(\gamma + |\boldsymbol{r}|)\exp(-|\boldsymbol{r}|/\gamma), \quad \alpha(0) = \frac{1}{4\gamma}. \tag{4.4}$$

(b) Two-dimensions:

$$\alpha(|\boldsymbol{r}|) = \frac{1}{2\pi}\frac{|\boldsymbol{r}|}{2\gamma^3}K_1(|\boldsymbol{r}|/\gamma), \qquad \alpha(0) = \frac{1}{4\pi\gamma^2}. \tag{4.5}$$

(c) Three-dimensions:

$$\alpha(|\boldsymbol{r}|) = \frac{1}{4\pi}\frac{1}{2\gamma^3}\exp(-|\boldsymbol{r}|/\gamma), \quad \alpha(0) = \frac{1}{8\pi\gamma^3}. \tag{4.6}$$

The kernels (4.1)–(4.6) fulfill all conditions (i)–(v). It is important to note that all kernels (4.1)–(4.6) are nonsingular in contrast to the two- and three-dimensional nonlocal kernels of Helmholtz type which are singular at $\boldsymbol{r} = 0$. In the limits $c_2 \to 0$ and $c_1 \to \varepsilon$, the nonlocal kernels (4.1)–(4.3) reduce to the nonlocal kernels of Helmholtz type given by Eringen (1983, 1987). The kernels (4.4)–(4.6) are plotted in Fig. 1.

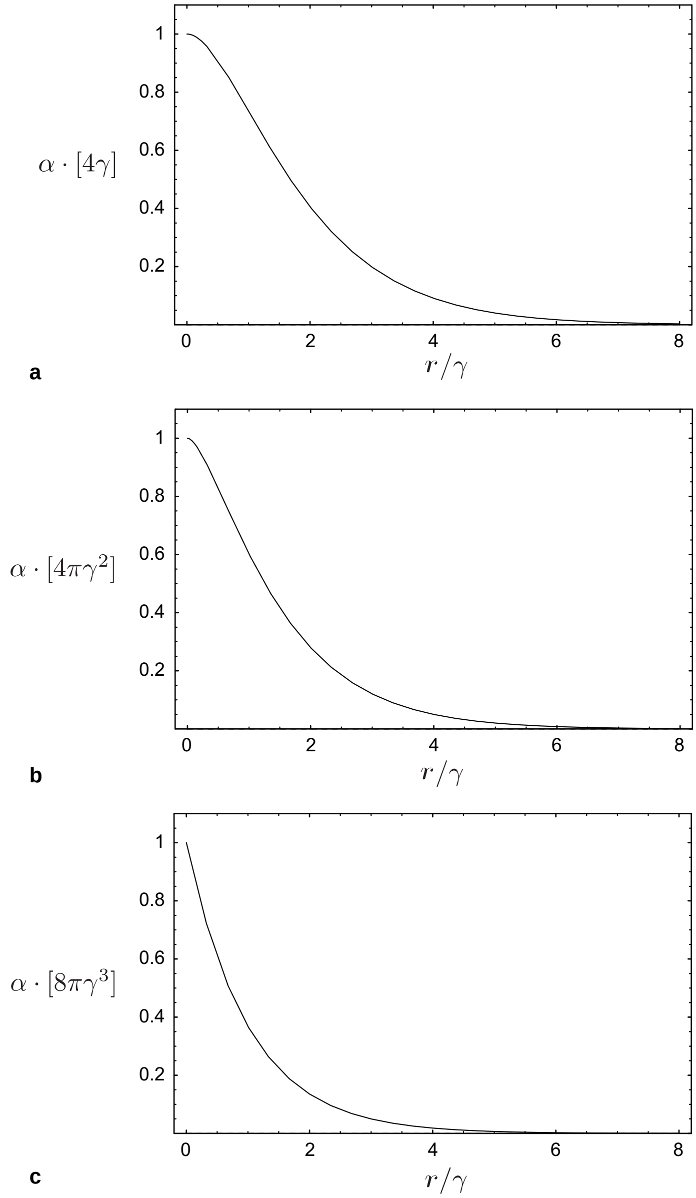

Fig. 1. Nonlocal kernels of bi-Helmholtz type with $c_1 = c_2 = \gamma$ are plotted vs r/γ: (a) α·[4γ], (b) α·[4$\pi\gamma^2$], (c) α·[8$\pi\gamma^3$].

5. Matching the dispersion curve with lattice models

We consider an elastic body of infinite extent with no discontinuity or body forces. Using the Helmholtz decomposition of the displacement field u_j into the Lamé potentials (scalar potential ϕ, vector potential ψ_j) according to

$$u_j = \partial_j \phi + \varepsilon_{jkl} \partial_k \psi_l \quad \text{with } \partial_j \psi_j = 0. \tag{5.1}$$

Eq. (3.6) with $f = 0$ reduces to

$$v_1^2 \Delta\phi - (1 - \varepsilon^2\Delta + \gamma^4\Delta\Delta)\ddot{\phi} = 0, \tag{5.2}$$

$$v_2^2 \Delta\psi - (1 - \varepsilon^2\Delta + \gamma^4\Delta\Delta)\ddot{\psi} = 0, \tag{5.3}$$

where

$$v_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad v_2^2 = \frac{\mu}{\rho}. \tag{5.4}$$

Here v_1 and v_2 denote the longitudinal and the transversal velocities of sound, respectively. For plane harmonic waves, Eqs. (5.2) and (5.3) lead to the following dispersion relations for the plane longitudinal and transverse waves

$$\omega_j^2(k)/\omega_{0j}^2 = \tilde{\alpha}(k) = \frac{1}{1 + \varepsilon^2k^2 + \gamma^4k^4}, \quad j = 1, 2 \tag{5.5}$$

with

$$\omega_{0j}^2 = k^2 v_j^2. \tag{5.6}$$

The phase velocity is given by

$$v_j^{\mathrm{P}}(k) = \omega_j(k)/k = v_j\sqrt{\tilde{\alpha}(k)} \tag{5.7}$$

and the group velocity reads

$$v_j^{\mathrm{G}}(k) = \frac{\mathrm{d}\omega_j}{\mathrm{d}k} = v_j \frac{1 - \gamma^4k^4}{\sqrt[3]{1 + \varepsilon^2k^2 + \gamma^4k^4}}. \tag{5.8}$$

The natural conditions for dispersion relations are

$$v_j^{\mathrm{G}}(0) = \left.\frac{\mathrm{d}\omega_j}{\mathrm{d}k}\right|_{k=0} = v_j, \quad v_j^{\mathrm{G}}(\pi/a) = \left.\frac{\mathrm{d}\omega_j}{\mathrm{d}k}\right|_{k=\frac{\pi}{a}} = 0. \tag{5.9}$$

The first condition means that the group velocity is v_j at the beginning of the Brillouin zone and the second condition of Eq. (5.9) means that the group velocity vanishes at the end of the Brillouin zone. From the second condition of (5.9), we obtain the value for γ_0 (see also Eringen, 2002):

$$\gamma_0 = \frac{1}{\pi} \simeq 0.318. \tag{5.10}$$

The dispersion relation (5.5) may be compared with those known from lattice dynamics to obtain the value of the coefficient ε_0.

5.1. Nearest neighbour interactions

In the Born–von Kármán model of lattice dynamics, which is analogous to the case of a chain with only nearest–neighbour interactions, the dispersion relation reads (see, e.g., Brillouin, 1953)

$$\omega_j^2(k)/\omega_{0j}^2 = (2a/k)^2\sin^2(ka/2). \tag{5.11}$$

The matching for (5.11) and (5.5) with (5.10) is perfect at the end of the Brillouin zone ($ka = \pi$) if (see Fig. 2)

$$\varepsilon_0 = \sqrt{\frac{\pi^2 - 8}{4\pi^2}} \simeq 0.218. \tag{5.12}$$

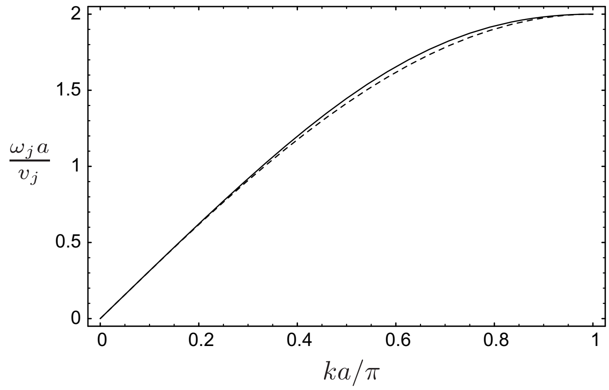

Fig. 2. Dispersion curves for nonlocal elasticity of bi-Helmholtz type with $\gamma_0 = 1/\pi$ and $\varepsilon_0 = 0.218$ (full line) and the Born–von Kármán model (dashed curve) of lattice dynamics vs reduced wave number ka/π.

The dispersion relation (5.5) with the values (5.10) and (5.12) coincides with Kunin's result for a Debye quasicontinuum (Kunin, 1983; Eringen, 2002). Unfortunately, it violates the condition (3.12). In fact:

$$\left(1 - 4\frac{\gamma_0}{\varepsilon_0}\right) < 0. \tag{5.13}$$

Thus, c_1 and c_2 are complex,

$$c_1 \simeq (0.250 + 0.197\mathrm{i})a, \tag{5.14}$$

$$c_2 \simeq (0.250 - 0.197\mathrm{i})a \tag{5.15}$$

with $\mathrm{i} = \sqrt{-1}$. But, this is not acceptable from the physical point of view.

If we use the case $\varepsilon^2 = 2\gamma^2$, the dispersion relation reads

$$\omega_j^2/\omega_{0j}^2 = \frac{1}{1 + 2\gamma^2 k^2 + \gamma^4 k^4}. \tag{5.16}$$

From the matching at $ka = \pi$, we obtain for γ_0:

$$\gamma_0 = \frac{\sqrt{2\pi - 4}}{2\pi} \simeq 0.240, \tag{5.17}$$

or for c_1 and c_2:

$$c_1 = c_2 \simeq 0.240a. \tag{5.18}$$

But the price one has to pay is that the group velocity is off at the end of the Brillouin zone. In fact, the group velocity is close to zero at the Brillouin zone boundary. However, it is a better approximation than the dispersion curve of nonlocal elasticity of Helmholtz type (see Fig. 3)

$$\omega_j^2/\omega_{0j}^2 = \frac{1}{1 + \varepsilon^2 k^2}, \quad \text{with} \quad \varepsilon_0 = 0.39, \tag{5.19}$$

where the group velocity is badly off at the end of the Brillouin zone.

Therefore, the best physical result for the match of the Born–von Kármán model is given by nonlocal elasticity of Helmholtz type or bi-Helmholtz type with one parameter of nonlocality. However, in both cases the group velocity is badly off at the end of the Brillouin zone.

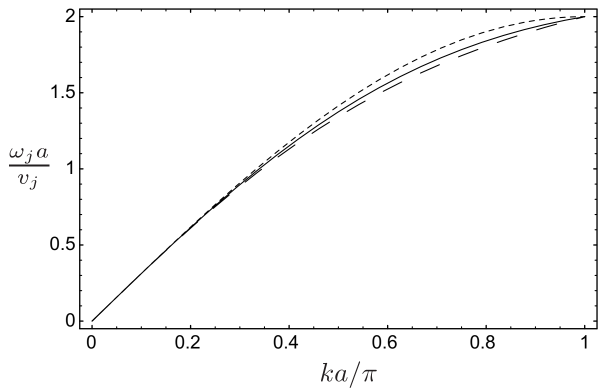

Fig. 3. Dispersion curves for nonlocal elasticity of bi-Helmholtz type with $\gamma_0 = 0.24$ (full line), the nonlocal model of Helmholtz type with $\varepsilon_0 = 0.39$ (long dashed curve) and the Born–von Kármán model (dashed curve) vs reduced wave number ka/π.

5.2. Next nearest neighbour interactions

Now, we consider a model of lattice dynamics with next-nearest neighbour interactions. Thus, we account for the interactions with first and second neighbours like in a homogeneous chain. The dispersion relation of such a model is given by Jaunzemis (1967).

$$\omega_j^2(k)/\omega_{0j}^2 = (2a/k)^2(1+\delta)\sin^2(ka/2) \tag{5.20}$$

with

$$\delta = \frac{\beta}{\alpha}, \tag{5.21}$$

where α and β are the force constants (stiffness) of first and second neighbour interactions. Thus, δ is the ratio of force constants for second and first neighbour interactions.

The matching for (5.20) and (5.5) with (5.10) is perfect at the end of the Brillouin zone ($ka = \pi$) if (see Fig. 4)

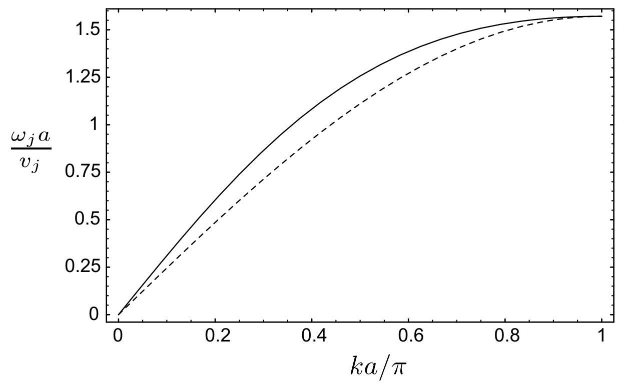

Fig. 4. Dispersion curves for nonlocal elasticity of bi-Helmholtz type with $\gamma_0 = 1/\pi$ and the lattice model with next-nearest neighbour interactions vs reduced wave number ka/π for the value of $\delta = -0.383$.

$$\varepsilon_0 = \sqrt{\frac{\pi^2 - 8(1+\delta)}{4\pi^2(1+\delta)}}. \tag{5.22}$$

Thus, ε_0 is given in terms of the ratio δ. If we use (5.22) and solve the inequation (3.12), we obtain the bounds for δ:

$$-1 < \delta \leqslant \frac{\pi^2}{16} - 1 \simeq -0.383 \tag{5.23}$$

and for the force constants

$$-\alpha < \beta \leqslant -0.383\alpha. \tag{5.24}$$

Thus, this corresponds to a competition between the interaction of second and first neighbours. The first force constant is stronger than the second one; otherwise, it would be an unphysical result. The sign and the bounds of δ are similar to those given by Maugin (1999, p. 73).

If we give up the second condition of (5.9) and match the dispersion relation (5.16) with (5.20) at $ka = \pi$, we obtain

$$\gamma_0^2 = -\frac{1}{\pi^2} + \sqrt{\frac{1}{4\pi^2(1+\delta)}}. \tag{5.25}$$

Doing the same for (5.19) and (5.20), one gets

$$\varepsilon_0^2 = \frac{\pi^2 - 4(1+\delta)}{4\pi^2(1+\delta)}. \tag{5.26}$$

From the conditions $\gamma_0^2 > 0$ and $\varepsilon_0^2 > 0$ we obtain in both cases the following bounds for δ:

$$-1 < \delta \leqslant \frac{\pi^2}{4} - 1 \simeq 1.467 \tag{5.27}$$

and for the force constants

$$-\alpha < \beta \leqslant 1.467\alpha. \tag{5.28}$$

Thus, $\beta > \alpha$ is possible, what is a strange and unphysical result.

We conclude that the best match for the lattice model with next-nearest neighbour interactions is obtained by the nonlocal elasticity of bi-Helmholtz type with two different coefficients of nonlocality. It predicts the physical result of a vanishing group velocity at the boundary of the Brillouin zone. No anomalous dispersion appears. Nonlocal elasticity of Helmholtz type or bi-Helmholtz type with only one parameter of nonlocality leads to unphysical results.

6. Screw dislocation

Consider now a static screw dislocation in the nonlocal theory of bi-Helmholtz type. For $f_i = 0$ the static equation of equilibrium (2.1) is satisfied by introducing the stress function F according to

$$t_{z\varphi} = \partial_r F. \tag{6.1}$$

For stress $\sigma_{z\varphi}$ of a straight screw dislocation we have

$$\sigma_{z\varphi} = \frac{\mu b_z}{2\pi}\frac{1}{r} = \frac{\mu b_z}{2\pi}\partial_r \ln r. \tag{6.2}$$

Using (3.4), this gives an inhomogeneous bi-Helmholtz equation for the stress function:

$$(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta)F=\frac{\mu b_z}{2\pi}\ln r. \tag{6.3}$$

An appropriate solution of this equation is given by (see Appendix A.2, Eq. (A.19)):

$$F=\frac{\mu b_z}{2\pi}\left\{\ln r+\frac{1}{c_1^2-c_2^2}\left[c_1^2K_0(r/c_1)-c_2^2K_0(r/c_2)\right]\right\}. \tag{6.4}$$

Hence, the stress field $t_{z\varphi}$ is given by

$$t_{z\varphi}=\frac{\mu b_z}{2\pi}\frac{1}{r}\left\{1-\frac{1}{c_1^2-c_2^2}\left[c_1rK_1(r/c_1)-c_2rK_1(r/c_2)\right]\right\}. \tag{6.5}$$

In the limits $c_2\to 0$ and $c_1\to\varepsilon$, we recover Eringen's result calculated in nonlocal elasticity of Helmholtz type (Eringen, 1983; Eringen, 2002). For $c_2=c_1=\gamma$ it reads

$$t_{z\varphi}=\frac{\mu b_z}{2\pi}\frac{1}{r}\left\{1-\frac{r}{\gamma}K_1(r/\gamma)-\frac{r^2}{2\gamma^2}K_0(r/\gamma)\right\}. \tag{6.6}$$

The stress is zero at $r=0$ and has an extremum value near the dislocation line. The extremum value depends strongly on c_2 and c_1. For $c_1=c_2=\gamma$, we have: $t_{z\varphi}\simeq 0.249\mu b_z/[2\pi\gamma]=0.352\mu b_z/[2\pi\varepsilon]$ at $r\simeq 2.324\gamma=1.643\varepsilon$. Eq. (6.6) is plotted over r/ε in Fig. 5. The stress in nonlocal elasticity of Helmholtz type has a maximum: $t_{z\varphi}\simeq 0.399\ \mu b_z/[2\pi\varepsilon]$ at $r\simeq 1.114\varepsilon$. Only in the region $r/\varepsilon<3$ is a difference between the stress calculated in nonlocal elasticity of Helmholtz or bi-Helmholtz type (see Fig. 5).

The stored strain energy is given by

$$\Sigma=\frac{1}{2}\int_V t_{ij}e_{ij}\,\mathrm{d}v, \tag{6.7}$$

where V is the volume of the solid body. Integrating over the region $r_0\leqslant r\leqslant R$, $0\leqslant\varphi\leqslant 2\pi$, $0\leqslant z\leqslant l$ and using $e_{z\varphi}=b_z/[4\pi r]$, one obtains the strain energy of a screw dislocation per length l as follows:

$$\frac{\Sigma_s}{l}=2\pi\int_{r_0}^{R}t_{z\varphi}e_{z\varphi}r\,\mathrm{d}r=\frac{\mu b_z^2}{4\pi}\left[\ln r+\frac{1}{c_1^2-c_2^2}\left[c_1^2K_0(r/c_1)-c_2^2K_0(r/c_2)\right]\right]\Bigg|_{r_0}^{R} \tag{6.8}$$

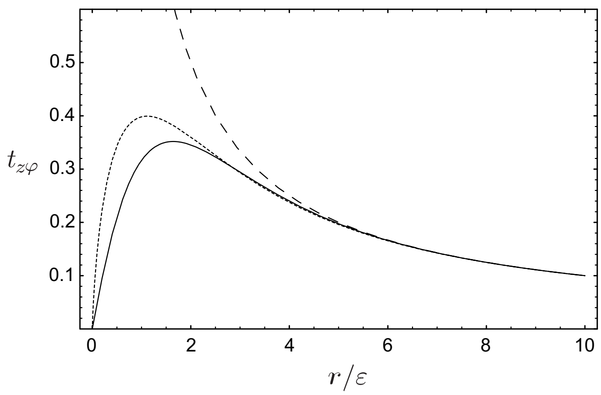

Fig. 5. Stress $t_{z\varphi}$ of a screw dislocation is given in units of $\mu b_z/[2\pi\varepsilon]$. The full curve, small dashed curve and dashed curve, respectively, represent the stress fields in nonlocal elasticity of bi-Helmholtz type, nonlocal elasticity of Helmholtz type and classical elasticity.

and thus

$$\frac{\Sigma_s}{l} = \frac{\mu b_z^2}{4\pi}\left[\ln\frac{R}{r_0} + \frac{1}{c_1^2 - c_2^2}\{c_1^2[K_0(R/c_1) - K_0(r_0/c_1)] - c_2^2[K_0(R/c_2) - K_0(r_0/c_2)]\}\right]. \tag{6.9}$$

With the limiting expression

$$K_0(r/c)|_{r\to 0} \;\to\; -\left[\gamma_{\mathrm{E}} + \ln\frac{r}{2c}\right], \tag{6.10}$$

where γ_{E} is the Euler constant, we find for a solid body of radius R:

$$\frac{\Sigma_s}{l} = \frac{\mu b_z^2}{4\pi}\left[\gamma_{\mathrm{E}} + \ln\frac{R}{2} + \frac{1}{c_1^2 - c_2^2}\{c_1^2[K_0(R/c_1) - \ln c_1] - c_2^2[K_0(R/c_2) - \ln c_2]\}\right]. \tag{6.11}$$

Unlike the classical result, Σ_s has no singularity as $r_0 \to 0$. Using

$$K_n(r/c)|_{r\to\infty} \;\to\; 0, \tag{6.12}$$

the final result reads

$$\frac{\Sigma_s}{l} = \frac{\mu b_z^2}{4\pi}\left[\gamma_{\mathrm{E}} + \ln\frac{R}{2} - \frac{1}{c_1^2 - c_2^2}\{c_1^2 \ln c_1 - c_2^2 \ln c_2\}\right]. \tag{6.13}$$

The strain energy depends on c_1 and c_2. In the limit $c_2 \to 0$, we recover the result of nonlocal elasticity of Helmholtz type (see, e.g., Eringen, 2002). The nonlocal result of Helmholtz type coincides with Gutkin and Aifantis' gradient elasticity result (Gutkin and Aifantis, 1999b) and, in addition, they agree with Lazar's gauge theoretical result (Lazar, 2002a; Lazar, 2002b), which explicitly contains the dislocation core energy. Thus, we conclude that the nonlocal strain energy is the total strain energy containing the core energy.

7. Edge dislocation

Now we consider an edge dislocation. In the case of a straight edge dislocation, the static equation of equilibrium (2.1) is fulfilled by using the stress function f according to

$$t_{rr} = \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial^2_{\varphi\varphi} f, \quad t_{\varphi\varphi} = \partial^2_{rr} f, \quad t_{r\varphi} = -\partial_r\left(\frac{1}{r}\partial_\varphi f\right), \quad t_{zz} = \nu(t_{rr} + t_{\varphi\varphi}). \tag{7.1}$$

The appropriate Airy stress function for a straight edge dislocation in classical elasticity is given by

$$\chi = -\frac{\mu b_x}{4\pi(1-\nu)}\partial_y(r^2 \ln r), \tag{7.2}$$

where ν is the Poisson ratio. The usual stress tensor is calculated by

$$\sigma_{rr} = \frac{1}{r}\partial_r\chi + \frac{1}{r^2}\partial^2_{\varphi\varphi}\chi \;= -\frac{\mu b_x}{2\pi(1-\nu)}\frac{\sin\varphi}{r}, \tag{7.3}$$

$$\sigma_{\varphi\varphi} = \partial^2_{rr}\chi = -\frac{\mu b_x}{2\pi(1-\nu)}\frac{\sin\varphi}{r}, \tag{7.4}$$

$$\sigma_{r\varphi} = -\partial_r\left(\frac{1}{r}\partial_\varphi\chi\right) = \frac{\mu b_x}{2\pi(1-\nu)}\frac{\cos\varphi}{r}. \tag{7.5}$$

If we use (3.4), we obtain an inhomogeneous bi-Helmholtz equation for the stress function:

$$(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta)f=-\frac{\mu b_x}{4\pi(1-\nu)}\partial_y(r^2\ln r). \tag{7.6}$$

The solution of this equation reads (see Appendix A.3, Eq. (A.25):

$$f=-\frac{\mu b_x}{2\pi(1-\nu)}\sin\varphi\left\{r\left(\ln r+\frac{1}{2}\right)+\frac{2(c_1^2+c_2^2)}{r}-\frac{2}{c_1^2-c_2^2}[c_1^3K_1(r/c_1)-c_2^3K_1(r/c_2)]\right\}. \tag{7.7}$$

We find for the stress of a straight edge dislocation in cylindrical coordinates

$$t_{rr}=-\frac{\mu b_x}{2\pi(1-\nu)}\frac{\sin\varphi}{r}\left\{1-\frac{4(c_1^2+c_2^2)}{r^2}+\frac{2}{c_1^2-c_2^2}[c_1^2K_2(r/c_1)-c_2^2K_2(r/c_2)]\right\}, \tag{7.8}$$

$$t_{r\varphi}=\frac{\mu b_x}{2\pi(1-\nu)}\frac{\cos\varphi}{r}\left\{1-\frac{4(c_1^2+c_2^2)}{r^2}+\frac{2}{c_1^2-c_2^2}[c_1^2K_2(r/c_1)-c_2^2K_2(r/c_2)]\right\}, \tag{7.9}$$

$$t_{\varphi\varphi}=-\frac{\mu b_x}{2\pi(1-\nu)}\frac{\sin\varphi}{r}\left\{1+\frac{4(c_1^2+c_2^2)}{r^2}-\frac{2}{c_1^2-c_2^2}[c_1^2K_2(r/c_1)-c_2^2K_2(r/c_2)+c_1rK_1(r/c_1)-c_2rK_1(r/c_2)]\right\}, \tag{7.10}$$

$$t_{zz}=-\frac{\mu b_x\nu}{\pi(1-\nu)}\frac{\sin\varphi}{r}\left\{1-\frac{1}{c_1^2-c_2^2}[c_1rK_1(r/c_1)-c_2rK_1(r/c_2)]\right\}. \tag{7.11}$$

In the limits $c_2\to 0$ and $c_1\to\varepsilon=1/\kappa$, we recover from Eqs. (7.8)–(7.11) the result of nonlocal elasticity of Helmholtz type obtained by Lazar (2003). In the limit $c_2\to c_2=\gamma$, we obtain

$$t_{rr}=-\frac{\mu b_x}{2\pi(1-\nu)}\frac{\sin\varphi}{r}\left\{1-\frac{8\gamma^2}{r^2}+4K_2(r/\gamma)+\frac{r}{\gamma}K_1(r/\gamma)\right\}, \tag{7.12}$$

$$t_{r\varphi}=\frac{\mu b_x}{2\pi(1-\nu)}\frac{\cos\varphi}{r}\left\{1-\frac{8\gamma^2}{r^2}+4K_2(r/\gamma)+\frac{r}{\gamma}K_1(r/\gamma)\right\}, \tag{7.13}$$

$$t_{\varphi\varphi}=-\frac{\mu b_x}{2\pi(1-\nu)}\frac{\sin\varphi}{r}\left\{1+\frac{8\gamma^2}{r^2}-4K_2(r/\gamma)-3\frac{r}{\gamma}K_1(r/\gamma)-\frac{r^2}{\gamma^2}K_0(r/\gamma)\right\}, \tag{7.14}$$

$$t_{zz}=-\frac{\mu b_x\nu}{\pi(1-\nu)}\frac{\sin\varphi}{r}\left\{1-\frac{r}{\gamma}K_1(r/\gamma)-\frac{r^2}{2\gamma^2}K_0(r/\gamma)\right\}. \tag{7.15}$$

The stresses are zero at $r=0$ and have extremum values near the dislocation line. The extremum values depend on c_2 and c_1. For $c_1=c_2=\gamma$, we have: $|t_{rr}|\simeq 0.159\mu\ b_x/[2\pi(1-\nu)\gamma]=0.225\mu b_x/[2\pi(1-\nu)\varepsilon]$ at $r\simeq 3.102$, $\gamma=2.193\varepsilon$ and $\varphi=\pi/2,\ 3\pi/2$, $|t_{r\varphi}|\simeq 0.159\mu b_x/[2\pi(1-\nu)\gamma]=0.225\mu\ b_x/[2\pi(1-\nu)\varepsilon]$ at $r\simeq 3.102\gamma=2.193\ \varepsilon$ and $\varphi=0,\ \pi$, $|t_{\varphi\varphi}|\simeq 0.345\mu\ b_x/[2\pi(1-\nu)\gamma]=0.489\mu b_x/[2\pi(1-\nu)\varepsilon]$ at $r\simeq 2.10\gamma=1.485\varepsilon$, and $\varphi=\pi/2,\ 3\pi/2$ and $|t_{zz}|\simeq 0.249\mu b_x/[\pi(1-\nu)\gamma]=0.352\mu\ b_x/[\pi(1-\nu)\varepsilon]$ at $r\simeq 2.324\gamma=1.643\varepsilon$ and $\varphi=\pi/2,\ 3\pi/2$. The stresses are plotted over r/ε in Fig. 6. It can be seen that the stresses calculated in nonlocal elasticity of bi-Helmholtz type are slightly changed in comparison to the corresponding stresses obtained in nonlocal elasticity of Helmholtz type which were calculated by Lazar (2003). In fact, in nonlocal elasticity of Helmholtz type the extremum values are: $|t_{rr}|\simeq 0.260\mu b_x/[2\pi(1-\nu)\varepsilon]$ at $r\simeq 1.494\varepsilon$ and $\varphi=\pi/2,\ 3\pi/2$, $|t_{r\varphi}|\simeq 0.260\mu\ b_x/[2\pi(1-\nu)\varepsilon]$ at $r\simeq 1.494\varepsilon$ and $\varphi=0,\ \pi$, $|t_{\varphi\varphi}|\simeq 0.547\mu\ b_x/[2\pi(1-\nu)\varepsilon]$ at $r\simeq 0.996\varepsilon$, and $\varphi=\pi/2,\ 3\pi/2$ and $|t_{zz}|\simeq 0.399\mu\ b_x/[\pi(1-\nu)\varepsilon]$ at $r\simeq 1.114\varepsilon$ and $\varphi=\pi/2,\ 3\pi/2$.

The stored strain energy (6.7) for an edge dislocation is given by

$$\Sigma_{\rm e}=\frac{1}{2}\int_V(t_{rr}e_{rr}+t_{\varphi\varphi}e_{\varphi\varphi}+2t_{r\varphi}e_{r\varphi})\,{\rm d}v. \tag{7.16}$$

Using (7.8)–(7.10) and e_{rr}, $e_{\varphi\varphi}$ and $e_{r\varphi}$ obtained from (7.3)–(7.5) with the inverse of the Hooke law (2.3) and integrating over the region $r_0\leqslant r\leqslant R$, $0\leqslant\varphi\leqslant 2\pi$, $0\leqslant z\leqslant l$, we find the strain energy per length l as

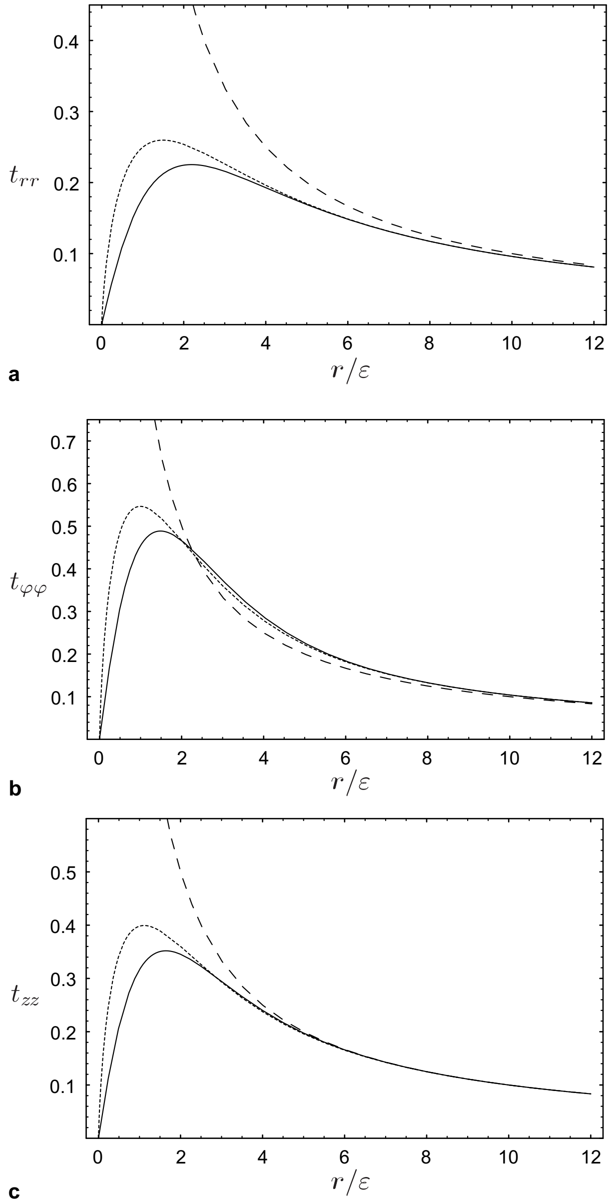

Fig. 6. Stress of an edge dislocation: (a) t_{rr} and (b) $t_{\varphi\varphi}$ are given in units of $\mu b_x/[2\pi(1-\nu)\varepsilon]$ for $\varphi = 3\pi/2$, (c) t_{zz} is given in units of $\mu b_x \nu/[\pi(1-\nu)\varepsilon]$ for $\varphi = 3\pi/2$. The full curves, small dashed curves and dashed curves, respectively, represent the stress fields in nonlocal elasticity of bi-Helmholtz type, nonlocal elasticity of Helmholtz type and classical elasticity.

$$\frac{\Sigma_{\mathrm{e}}}{l} = \frac{\mu b_x^2}{4\pi(1-\nu)^2}\left[(1-\nu)\left(\ln r + \frac{1}{c_1^2 - c_2^2}\left[c_1^2 K_0(r/c_1) - c_2^2 K_0(r/c_2)\right]\right)\right.$$
$$\left. + \frac{c_1^2 + c_2^2}{r^2} - \frac{1}{2(c_1^2 - c_2^2)}\left[c_1^2 K_2(r/c_1) - c_2^2 K_2(r/c_2)\right]\right]\Bigg|_{r_0}^{R} \quad (7.17)$$

and thus

$$\frac{\Sigma_{\rm e}}{l}=\frac{\mu b_x^2}{4\pi(1-\nu)^2}\left[(1-\nu)\left(\ln\frac{R}{r_0}+\frac{1}{c_1^2-c_2^2}\left\{c_1^2[K_0(R/c_1)-K_0(r_0/c_1)]-c_2^2[K_0(R/c_2)-K_0(r_0/c_2)]\right\}\right)\right.$$
$$\left.+\frac{c_1^2+c_2^2}{R^2}-\frac{c_1^2+c_2^2}{r_0^2}-\frac{1}{2(c_1^2-c_2^2)}\left\{c_1^2[K_2(R/c_1)-K_2(r_0/c_1)]-c_2^2[K_2(R/c_2)-K_2(r_0/c_2)]\right\}\right]. \quad (7.18)$$

By using the relation (6.10) as well as the limiting expression

$$K_2(r/c)|_{r\to 0}\ \to\ -\frac{1}{2}+\frac{2c^2}{r^2}, \quad (7.19)$$

we find the elastic energy for a solid with radius R:

$$\frac{\Sigma_{\rm e}}{l}=\frac{\mu b_x^2}{4\pi(1-\nu)^2}\left[(1-\nu)\left(\gamma_{\rm E}+\ln\frac{R}{2}+\frac{1}{c_1^2-c_2^2}\{c_1^2[K_0(R/c_1)-\ln c_1]-c_2^2[K_0(R/c_2)-\ln c_2]\}\right)\right.$$
$$\left.-\frac{1}{4}+\frac{c_1^2+c_2^2}{R^2}-\frac{1}{2(c_1^2-c_2^2)}\{c_1^2K_2(R/c_1)-c_2^2K_2(R/c_2)\}\right]. \quad (7.20)$$

With the asymptotic formula (6.12), we finally obtain

$$\frac{\Sigma_{\rm e}}{l}=\frac{\mu b_x^2}{4\pi(1-\nu)}\left[\gamma_{\rm E}-\frac{1}{4(1-\nu)}+\ln\frac{R}{2}-\frac{1}{c_1^2-c_2^2}\{c_1^2\ln c_1-c_2^2\ln c_2\}\right]. \quad (7.21)$$

Again, the strain energy depends on c_1 and c_2. It can be seen that the energy (7.21) is similar to the result for the energy of a screw dislocation. Only the pre-factor and the second piece of (7.21) are different. In the limit $c_2 \to 0$ (nonlocal elasticity of Helmholtz type) we recover the result obtained by Gutkin and Aifantis calculated in the theory of gradient elasticity (Eq. (27) in Gutkin and Aifantis (1999b)). Gutkin and Aifantis (1999b) used in gradient elasticity the strain energy $W=\frac{1}{2}\int_V\sigma_{ij}\epsilon_{ij}^{(\rm gr)}\,{\rm d}v$ where $\epsilon_{ij}^{(\rm gr)}$ is the elastic strain calculated in gradient elasticity. Gutkin and Aifantis (1999b) also calculated the strain energy of a screw and an edge dislocation with another expression for the (elastic) strain energy $W=-\frac{1}{2}\int_V\sigma_{ij}^{(\rm gr)}\beta_{ij}^{\rm P,(cl)}\,{\rm d}v$ where $\sigma_{ij}^{(\rm gr)}$ denotes the nonsingular stress in gradient elasticity and $\beta_{ij}^{\rm P,(cl)}$ is the classical plastic distortion tensor. Because the elastic strain energy should be given only in terms of elastic fields, the plastic distortion should not enter the elastic strain energy. The price they had to pay is that for the edge dislocation they obtained a different result which does not agree with the elastic strain energy calculated in nonlocal elasticity of Helmholtz type.

8. Conclusions

In this paper, nonlocal elasticity of bi-Helmholtz type is studied. New nonlocal kernels in one-, two-, and three-dimensions are calculated. These kernels contain two parameters of nonlocality. All new nonlocal kernels are nonsingular. Dispersion relations are obtained for plane waves in nonlocal elasticity of bi-Helmholtz type. By equating the frequency at the end of the Brillouin zone, one parameter of nonlocality is determined. The second one is determined by the condition that the group velocity is zero at the boundary of the Brillouin zone. We compared the dispersion curves with results known from lattice dynamics with nearest neighbour interactions and with next nearest neighbour interactions. The best physical result for the dispersion relation of nonlocal elasticity of bi-Helmholtz type with two different parameters is obtained from lattice theory with next nearest neighbour interactions. In addition, the stresses and strain energy of a straight screw dislocation as well as an edge dislocation are calculated. They do not have singularities. The corresponding static theory of gradient elasticity of bi-Helmholtz type with double and triple stresses will be given in a further publication (Lazar et al., submitted for publication).

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Appendix A

We use the following notation for the Fourier transform (Guelfand and Chilov, 1962)

$$\widetilde{f}(\boldsymbol{k}) \equiv \mathscr{F}_{(n)}[f(\boldsymbol{r})] = \int_{-\infty}^{+\infty} f(\boldsymbol{r})\mathrm{e}^{+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{k}, \tag{A.1}$$

$$f(\boldsymbol{r}) \equiv \mathscr{F}^{-1}_{(n)}[\widetilde{f}(\boldsymbol{k})] = \frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty} \widetilde{f}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{r}. \tag{A.2}$$

A.1. Nonlocal kernels of the bi-Helmholtz equation

First we want to find the nonlocal kernels by using the Fourier transform. From the bi-Helmholtz equation

$$(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta)\alpha(|\boldsymbol{r}|) = \delta(\boldsymbol{r}), \tag{A.3}$$

the Fourier transformed nonlocal kernel, which is the Green function, is given by

$$\tilde{\alpha}(k) = \frac{1}{1+\varepsilon^2k^2+\gamma^4k^4}. \tag{A.4}$$

The calculation of $\mathscr{F}^{-1}_{(n)}[\tilde{\alpha}(k)]$ is depending on the zero points of the polynomial in k^2:

$$k^4+\frac{\varepsilon^2}{\gamma^4}k^2+\frac{1}{\gamma^4}. \tag{A.5}$$

The zero points are given by

$$k^2_{1/2} = -\frac{\varepsilon^2}{2\gamma^4}\left(1\mp\sqrt{1-4\frac{\gamma^4}{\varepsilon^4}}\right). \tag{A.6}$$

Therefore, a necessary condition for the calculation of the inverse Fourier transform is that the discriminant has to be nonnegative. Therefore,

$$0 \leqslant \left(1-4\frac{\gamma^4}{\varepsilon^4}\right), \tag{A.7}$$

which is equivalent with the condition (3.12). Now we factorize Eq. (A.4) and use the abbreviations (3.8) and (3.9) according to

$$\tilde{\alpha}(k) = \frac{1}{(1+c_1^2k^2)(1+c_2^2k^2)} = \frac{1}{c_1^2-c_2^2}\left(\frac{1}{k^2+\frac{1}{c_1^2}}-\frac{1}{k^2+\frac{1}{c_2^2}}\right). \tag{A.8}$$

Using the formulas (Guelfand and Chilov, 1962; Wladimirow, 1971)

$$\mathscr{F}^{-1}_{(1)}\left[\frac{1}{k^2+\frac{1}{c^2}}\right]=\frac{c}{2}\exp(-|x|/c) \tag{A.9}$$

$$\mathscr{F}^{-1}_{(2)}\left[\frac{1}{k^2+\frac{1}{c^2}}\right]=\frac{1}{2\pi}K_0(\sqrt{x^2+y^2}/c) \tag{A.10}$$

$$\mathscr{F}^{-1}_{(3)}\left[\frac{1}{k^2+\frac{1}{c^2}}\right]=\frac{1}{4\pi\sqrt{x^2+y^2+z^2}}\exp(-\sqrt{x^2+y^2+z^2}/c) \tag{A.11}$$

we obtain the inverse Fourier transformed nonlocal kernels

$$1\text{D}:\alpha(|\boldsymbol{r}|)=\frac{1}{2}\frac{1}{c_1^2-c_2^2}\{c_1\exp(-|\boldsymbol{r}|/c_1)-c_2\exp(-|\boldsymbol{r}|/c_2)\},\quad |\boldsymbol{r}|=\sqrt{x^2} \tag{A.12}$$

$$2\text{D}:\alpha(|\boldsymbol{r}|)=\frac{1}{2\pi}\frac{1}{c_1^2-c_2^2}\{K_0(|\boldsymbol{r}|/c_1)-K_0(|\boldsymbol{r}|/c_2)\},\quad |\boldsymbol{r}|=\sqrt{x^2+y^2} \tag{A.13}$$

$$3\text{D}:\alpha(|\boldsymbol{r}|)=\frac{1}{4\pi}\frac{1}{c_1^2-c_2^2}\frac{1}{|\boldsymbol{r}|}\{\exp(-|\boldsymbol{r}|/c_1)-\exp(-|\boldsymbol{r}|/c_2)\},\quad |\boldsymbol{r}|=\sqrt{x^2+y^2+z^2}. \tag{A.14}$$

A.2. Stress function of bi-Helmholtz type for a screw dislocation

The stress function F fulfills the inhomogeneous bi-Helmholtz equation

$$(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta)F=\frac{A}{2\pi}\ln r,\quad A=\mu b_z \tag{A.15}$$

and is the Green function of the following PDE of sixth order (bi-Helmholtz Laplace equation):

$$(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta)\Delta F=A\delta(x)\delta(y). \tag{A.16}$$

Thus, we just calculate the Green function of Eq. (A.16). The Fourier transform of the Green function of (A.16) reads

$$\widetilde{F}(k)=-\frac{A}{k^2(1+\varepsilon^2k^2+\gamma^4k^4)}=-\frac{A}{k^2}+\frac{A}{c_1^2-c_2^2}\left(\frac{c_1^2}{k^2+\frac{1}{c_1^2}}-\frac{c_2^2}{k^2+\frac{1}{c_2^2}}\right). \tag{A.17}$$

Using Eq. (A.10) and the formula (Wladimirow, 1971)

$$\mathscr{F}^{-1}_{(2)}\left[\frac{1}{k^2}\right]=-\frac{1}{2\pi}(\gamma_{\text{E}}+\ln r), \tag{A.18}$$

the solution for F is given by

$$F=\frac{A}{2\pi}\left\{\ln r+\frac{1}{c_1^2-c_2^2}[c_1^2K_0(r/c_1)-c_2^2K_0(r/c_2)]\right\}. \tag{A.19}$$

The constant factor γ_{E} in (A.18) is not relevant for the Green function due to the Laplacian in (A.16).

A.3. Stress function of bi-Helmholtz type for an edge dislocation

We use the relation $f=\partial_y G$ and obtain for the stress function G the following inhomogeneous bi-Helmholtz equation

$$(1 - \varepsilon^2 \Delta + \gamma^4 \Delta\Delta) G = \frac{A}{4\pi} r^2 \ln r, \quad A = -\frac{\mu b_x}{1 - \nu}. \tag{A.20}$$

The stress function G is the Green function of the PDE of eighth order (bi-Helmholtz bi-Laplace equation):

$$(1 - \varepsilon^2 \Delta + \gamma^4 \Delta\Delta) \Delta\Delta G = 2A\delta(x)\delta(y). \tag{A.21}$$

Again we determine the stress function G as the Green function. From Eq. (A.21) we obtain for the Fourier transform of the Green function:

$$\widetilde{G}(k) = \frac{2A}{k^4(1 + \varepsilon^2 k^2 + \gamma^4 k^4)} = 2A \left[\frac{1}{k^4} - \frac{c_1^2 + c_2^2}{k^2} + \frac{1}{c_1^2 - c_2^2} \left(\frac{c_1^4}{k^2 + \frac{1}{c_1^2}} - \frac{c_2^4}{k^2 + \frac{1}{c_2^2}} \right) \right]. \tag{A.22}$$

If we use Eq. (A.10) and the relation

$$\mathscr{F}^{-1}_{(2)} \left[\frac{1}{k^4} \right] = \frac{1}{8\pi} r^2 (\gamma_{\mathrm{E}} + \ln r), \tag{A.23}$$

we finally find the solution for Green function G of Eq. (A.21):

$$G = \frac{A}{2\pi} \left\{ \frac{r^2}{2} \ln r + 2(c_1^2 + c_2^2) \ln r + \frac{2}{c_1^2 - c_2^2} [c_1^4 K_0(r/c_1) - c_2^4 K_0(r/c_2)] \right\} \tag{A.24}$$

and for f:

$$f = \frac{A}{2\pi} \sin\varphi \left\{ r \left(\ln r + \frac{1}{2} \right) + \frac{2(c_1^2 + c_2^2)}{r} - \frac{2}{c_1^2 - c_2^2} [c_1^3 K_1(r/c_1) - c_2^3 K_1(r/c_2)] \right\}. \tag{A.25}$$

The factor $r^2 \gamma_{\mathrm{E}}$ in (A.23) is irrelevant for the Green function G since the bi-Laplacian in (A.21).

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